

Nonlinear Instabilities, Negative Energy Modes and Generalized Cherry Oscillators

D. Pfirsch

Max-Planck-Institut für Plasmaphysik, EURATOM Association, Garching, F. R. Germany

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In 1925 Cherry [1] discussed two oscillators of positive and negative energy that are nonlinearly coupled in a special way, and presented a class of exact solutions of the nonlinear equations showing explosive instability independent of the strength of the nonlinearity and the initial amplitudes. In this paper Cherry's Hamiltonian is transformed into a form which allows a simple physical interpretation. The new Hamiltonian is generalized to three nonlinearly coupled oscillators; it corresponds to three-wave interaction in a continuum theory, like the Vlasov-Maxwell theory, if there exist linear negative energy waves [2–4, 5, 6]. Cherry was able to present a two-parameter solution set for his example which would, however, allow a four-parameter solution set, and, as a first result, an analogous three-parameter solution set for the resonant three-oscillator case is obtained here which, however, would allow a six-parameter solution set. Nonlinear instability is therefore proven so far only for a very small part of the phase space of the oscillators. This paper gives in addition the complete solution for the three-oscillator case and shows that, except for a singular case, all initial conditions, especially those with arbitrarily small amplitudes, lead to explosive behaviour. This is true of the resonant case. The non-resonant oscillators can sometimes also become explosively unstable, but the initial amplitudes must not be infinitesimally small. A few examples are presented for illustration.

1. Cherry's Oscillators

In 1925 Cherry [1] discussed nonlinearly coupled oscillators which are described by the Hamiltonian

$$H = -\frac{1}{2} \omega_1 (p_1^2 + q_1^2) + \frac{1}{2} \omega_2 (p_2^2 + q_2^2) + \frac{\alpha}{2} (2q_1 p_1 p_2 - q_2 (q_1^2 - p_1^2)). \quad (1)$$

The constant α measures the effect of nonlinearity. For $\alpha=0$ one has two uncoupled oscillators of frequencies $\omega_1 > 0$ and $\omega_2 > 0$ which possess negative and positive energy, respectively. If $\omega_2 = 2\omega_1$, one has a third-order resonance. Cherry found for this case the following exact two-parameter solution set:

$$\begin{aligned} q_1 &= \frac{\sqrt{2}}{\varepsilon - \alpha t} \sin(\omega_1 t + \gamma), & p_1 &= \frac{-\sqrt{2}}{\varepsilon - \alpha t} \cos(\omega_1 t + \gamma), \\ q_2 &= \frac{-1}{\varepsilon - \alpha t} \sin(2\omega_1 t + 2\gamma), \\ p_2 &= \frac{-1}{\varepsilon - \alpha t} \cos(2\omega_1 t + 2\gamma), \end{aligned} \quad (2)$$

Reprint requests to Prof. Dr. D. Pfirsch, Max-Planck-Institut für Plasmaphysik, EURATOM Association, D-8046 Garching bei München.

where ε and γ are determined by the initial conditions. These relations show explosive instability for any $\alpha \neq 0$, whereas the linearized theory gives only stable oscillations. There is also no threshold amplitude. Small initial amplitudes only mean that it takes a long time for the explosion to occur.

In a continuum theory, like the Vlasov-Maxwell theory, the assumed resonance corresponds to the conservation law

$$\omega_1 + \omega_2 + \omega_3 = 0 \quad (3)$$

for a three-wave interaction. It is therefore of interest to have a formulation and an example which are closer to the structure of a three-wave interaction. To this end we introduce complex quantities given by

$$\xi = p + i q, \quad \xi^* = p - i q. \quad (4)$$

We can do a canonical transformation to ξ^* as the new momentum and to $\xi/2i$ as the new coordinate. Cherry's Hamiltonian then becomes

$$H = -\frac{1}{2} \omega_1 \xi_1^* \xi_1 + \frac{1}{2} \omega_2 \xi_2^* \xi_2 + \frac{\alpha}{4i} (\xi_1^2 \xi_2 - \xi_1^{*2} \xi_2^*). \quad (5)$$

This exhibits a simple structure of the nonlinear term which also allows a simple physical interpretation: in

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quantum theoretical language it means the simultaneous annihilation or creation of two quanta of frequency ω_1 with energy $-\hbar \omega_1$ each and of one quantum of frequency ω_2 with energy $+\hbar \omega_2$. If $\omega_2 = 2\omega_1$ these processes leave the energy unchanged and therefore allow the amplitudes to grow. The same holds for the first two terms in H . The growth of the amplitudes is, of course, only possible for perturbations with vanishing H .

2. Generalization to Three Coupled Oscillators

In this paper we give a generalization of the two coupled oscillators described by the Hamiltonian (5) to three coupled oscillators corresponding to the mentioned three-wave interaction in a continuum. The Hamiltonian which will be investigated is

$$H = \frac{1}{2} \sum_{k=1}^3 \omega_k \xi_k^* \xi_k + \frac{1}{2} \alpha \xi_1 \xi_2 \xi_3 + \frac{1}{2} \alpha^* \xi_1^* \xi_2^* \xi_3^*. \quad (6)$$

Invariance to time reversal is guaranteed for purely imaginary α . The frequencies ω_k are assumed in this section to satisfy the three wave conservation law

$$\sum_{k=1}^3 \omega_k = 0. \quad (7)$$

The equations of motion corresponding to the Hamiltonian (6) are

$$\dot{\xi}_k = i \omega_k \xi_k + i \alpha^* \xi_1^* \xi_2^* \xi_3^* / \xi_k^*. \quad (8)$$

The ansatz

$$\xi_k(t) = a(t) e^{i\omega_k t + i\varphi_k}, \quad \sum_{k=1}^3 \varphi_k = 0 \quad (9)$$

with $a(t)$ independent of k leads to the following equation for this quantity:

$$\dot{a} = i \alpha^* a^{*2}. \quad (10)$$

This can be solved by

$$a = \gamma b(t), \quad \gamma = \left(\frac{i \alpha^*}{|\alpha|^4} \right)^{1/3}, \quad b^* = b \quad (11)$$

and

$$\dot{b} = b^2. \quad (12)$$

This equation has the general solution

$$b = \frac{1}{\hat{\varepsilon} - t} \quad (13)$$

with $\hat{\varepsilon}$ being a constant of integration. We therefore obtained a three-parameter solution set

$$\xi_k = \left(\frac{i \alpha^*}{|\alpha|} \right)^{1/3} \frac{1}{\varepsilon - |\alpha|t} e^{i\omega_k t + i\varphi_k}, \quad \sum_{k=1}^3 \varphi_k = 0, \quad (14)$$

where

$$\varepsilon = \hat{\varepsilon} / |\alpha|. \quad (15)$$

These solutions correspond to Cherry's two-parameter solution set.

3. Reduction of the Equations of Motion in the Three-Oscillator Case

The equations of motion (8) can be written as

$$\dot{\xi}_k = i \omega_k \xi_k + f^* / \xi_k^*, \quad (16)$$

where

$$f^* = i \alpha^* \xi_1^* \xi_2^* \xi_3^* \quad (17)$$

is independent of the special oscillator. From (16) one finds

$$\frac{d}{dt} \xi_k \xi_k^* = f + f^*. \quad (18)$$

If one defines

$$F = \int_0^t f dt', \quad (19)$$

one obtains from (18)

$$\xi_k \xi_k^* = \lambda_k + F + F^*. \quad (20)$$

λ_k are real positive constants:

$$\lambda_k \geq 0. \quad (21)$$

Equation (16) then has the formal solution

$$\xi_k = \xi_k^0 e^{G_k} \quad (22)$$

with

$$G_k = i \omega_k t + \int_0^t dt' \frac{f^*}{\lambda_k + F + F^*}. \quad (23)$$

When f is decomposed into its real and imaginary parts:

$$f = f_R + i f_I, \quad (24)$$

one can write G_k as

$$G_k = i \omega_k t - i \int_0^t dt' \frac{f_I}{\lambda_k + 2F_R} + \frac{1}{2} \ln \left(1 + \frac{2F_R}{\lambda_k} \right). \quad (25)$$

The definition (17) for f then yields the relation

$$f = -i \alpha \prod_{k=1}^3 \xi_k^0 e^{G_k}. \quad (26)$$

From this relation one finds

$$|f|^2 = |f^0|^2 \prod_{k=1}^3 \left(1 + \frac{2F_R}{\lambda_k}\right) \quad (27)$$

and

$$\dot{f} = \sum_{k=1}^3 \left(i \omega_k + \frac{f^*}{\lambda_k + 2F_R} \right) f. \quad (28)$$

It is helpful to decompose this equation into its real and imaginary parts:

$$\dot{f}_R = \sum_{k=1}^3 \left(-\omega_k f_I + \frac{f f^*}{\lambda_k + 2F_R} \right), \quad (29)$$

$$\dot{f}_I = \sum_{k=1}^3 \omega_k f_R. \quad (30)$$

4. The Resonant case $\sum_{k=1}^3 \omega_k = 0$

In this case one has

$$f_I = \text{const} \quad (31)$$

and

$$\begin{aligned} \dot{f}_R &= |f^0|^2 \sum_{k=1}^3 \frac{1}{\lambda_k + 2F_R} \prod_{l=1}^3 \left(1 + \frac{2F_R}{\lambda_l}\right) \\ &= |f^0|^2 \left(\frac{1}{\lambda_3} \left(1 + \frac{2F_R}{\lambda_1}\right) \left(1 + \frac{2F_R}{\lambda_2}\right) \right. \\ &\quad \left. + \frac{1}{\lambda_1} \left(1 + \frac{2F_R}{\lambda_2}\right) \left(1 + \frac{2F_R}{\lambda_3}\right) \right. \\ &\quad \left. + \frac{1}{\lambda_2} \left(1 + \frac{2F_R}{\lambda_3}\right) \left(1 + \frac{2F_R}{\lambda_1}\right) \right) \\ &= |f^0|^2 \left[\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right. \\ &\quad \left. + 4F_R \left(\frac{1}{\lambda_1 \lambda_2} + \frac{1}{\lambda_2 \lambda_3} + \frac{1}{\lambda_3 \lambda_1} \right) \right. \\ &\quad \left. + (F_R)^2 \frac{12}{\lambda_1 \lambda_2 \lambda_3} \right], \end{aligned} \quad (32)$$

where

$$|f^0|^2 = |\alpha|^2 \lambda_1 \lambda_2 \lambda_3. \quad (33)$$

Since all the λ 's are non-negative, the r.h.s. can vanish only for negative values of F_R . The zeros are

$$\begin{aligned} F_{R\pm} &= -\frac{1}{6} \sum_{k=1}^3 \lambda_k \\ &\pm \frac{1}{6\sqrt{2}} \sqrt{(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + \lambda_3 - \lambda_1)^2}. \end{aligned} \quad (34)$$

According to definition (19) F_R is zero at $t=0$. Negative values of F_R are therefore obtained only for initially negative $\dot{F}_R = f_R$. But also these initial conditions lead to $F_R > 0$ after some time except in the singular case, leading to

$$F_R(t \rightarrow \infty) = F_{R+}. \quad (35)$$

For all other initial conditions the asymptotic behaviour is characterized by the $(F_R)^2$ term on the r.h.s. of (32) becoming the dominant one. First the once-integrated form of (32) is written down:

$$\begin{aligned} \frac{1}{2} (f_R)^2 &= \frac{1}{2} (f_R^0)^2 \\ &+ |f^0|^2 \left[\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) F_R \right. \\ &\quad \left. + 2F_R^2 \left(\frac{1}{\lambda_1 \lambda_2} + \frac{1}{\lambda_2 \lambda_3} + \frac{1}{\lambda_3 \lambda_1} \right) \right. \\ &\quad \left. + 4F_R^3 \frac{1}{\lambda_1 \lambda_2 \lambda_3} \right]. \end{aligned} \quad (36)$$

Its asymptotic form is

$$f_R^2 = |f^0|^2 \frac{8}{\lambda_1 \lambda_2 \lambda_3} F_R^3 = 8|\alpha|^2 F_R^3. \quad (37)$$

The solution of this equation with the constant of integration t_0 for $t < t_0$ is

$$F_R = \frac{1}{2|\alpha|^2} \frac{1}{(t-t_0)^2}, \quad (38)$$

from which it follows that

$$f_R = \frac{1}{|\alpha|^2} \frac{1}{(t-t_0)^3}. \quad (39)$$

Equation (25) then yields for t close to t_0

$$G_k = i \omega_k t + \frac{1}{2} \ln \frac{1}{\lambda_k |\alpha|^2 (t-t_0)^2} + \text{const} \quad (40)$$

and therefore

$$\xi_k \propto \frac{e^{i\omega_k t}}{t-t_0}. \quad (41)$$

The general solution of (32) or (36), valid for all times except at *reflection* points, can be expressed in terms of the Weierstrass \wp function:

$$F_R = \frac{1}{2|\alpha|^2} \wp(t-t_0, g_2, g_3) - \frac{B}{3A}, \quad (42)$$

where g_2 and g_3 are the so-called invariants, which in the present case are given by

$$g_2 = \frac{1}{12} B^2 - \frac{1}{4} C A \quad (43)$$

and

$$g_3 = \frac{1}{48} A B C - \frac{1}{16} D A^2 - \frac{2}{16 \cdot 27} C^3. \quad (44)$$

The quantities A, B, C, D are

$$\begin{aligned} A &= 8|\alpha|^2, \\ B &= 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ C &= 2|\alpha|^2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1), \\ D &= (f_R^0)^2. \end{aligned} \quad (45)$$

These are the coefficients in (36) when this equation is written as

$$(f_R)^2 = D + C F_R + B F_R^2 + A F_R^3. \quad (46)$$

The constant of integration t_0 is obtained from $F_R(t=0)=0$:

$$\frac{1}{2|\alpha|^2} \wp(-t_0, g_2, g_3) = \frac{B}{3A}. \quad (47)$$

It is necessarily real. Since the Weierstrass \wp function is an even function, t_0 satisfying (47) can have either sign. The sign is determined by

$$\dot{F}_R(t=0) = f_R^0. \quad (48)$$

After a *reflection* at a time t_R , the time t runs backward corresponding to the other possible sign of f_R in (46), i.e. the solution is then the one which is obtained by the replacement

$$\wp(t-t_0) \rightarrow \wp(2t_R-t_0-t). \quad (49)$$

Furthermore, \wp is a doubly-periodic meromorphic function in the complex $t-t_0$ plane, and since g_2 and g_3 are real, it is periodic along the real $t-t_0$ axis. If this

property is combined with the fact that \wp possesses as the only singularities a double pole at vanishing argument and corresponding periodic points, one always finds the behaviour exhibited by (38). The special solutions corresponding to (35) imply parameters g_2, g_3 which lead to an infinitely long period of \wp along the real axis. Also Cherry's solutions for the two-oscillator case and those for the three-oscillator case presented in Sect. 2 belong to this class.

In order to write down the ξ_k 's, one has to do the integral occurring in (25), which can be expressed in terms of Weierstrass's ζ and σ functions. This will, however, not be done here, since the main emphasis is on the time dependence of the amplitudes.

5. The non-resonant case $\sum_{k=1}^3 \omega_k \neq 0$

With

$$\sum_{k=1}^3 \omega_k = \Omega \quad (50)$$

one obtains from (29)

$$\dot{f}_R = -\Omega \dot{f}_I + \frac{d}{dt} \sum_{k=1}^3 \frac{f f^*}{\lambda_k + 2F_R}, \quad (51)$$

which, by means of (30), becomes

$$\dot{f}_R = -\Omega^2 \dot{F}_R + \frac{d}{dt} \sum_{k=1}^3 \frac{f f^*}{\lambda_k + 2F_R}. \quad (52)$$

Integration over t with $f_I^0 = f_I(t=0)$ then yields

$$\dot{f}_R = -\Omega^2 F_R + \sum_{k=1}^3 \frac{f f^*}{\lambda_k + 2F_R} - \Omega f_I^0, \quad (53)$$

which replaces (32). The r.h.s. of this equation is again a polynomial of second order in F_R :

$$\dot{f}_R = c + b F_R + a F_R^2 \quad (54)$$

with

$$\begin{aligned} a &= 12|\alpha|^2, \\ b &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ c &= -\Omega f_I^0 + |\alpha|^2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1). \end{aligned} \quad (55)$$

Since a is positive, sufficiently large $F_R > 0$ always leads to runaway of this quantity. If the equation

$$c + b F_R + a F_R^2 = 0 \quad (56)$$

has no real solution, then one has $\dot{f}_R > 0$ for all F_R , which results in explosive behaviour for arbitrary F_R . The same is true if there are two real solutions that are

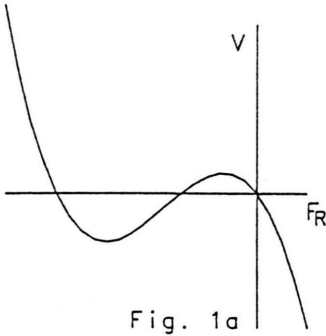


Fig. 1a

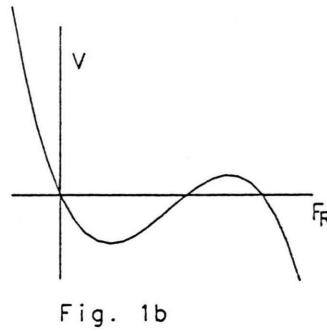


Fig. 1b

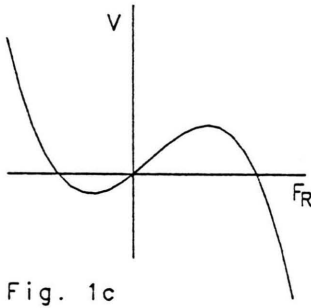


Fig. 1c

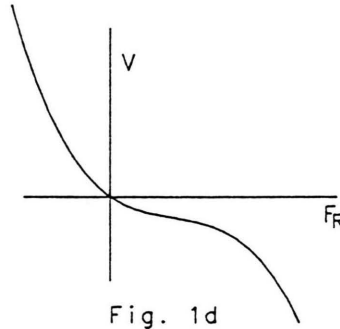


Fig. 1d

Fig. 1. Typical forms of the potential $V(F_R)$.

both negative. This was the situation with the resonant case. If at least one solution is positive, a threshold initial amplitude may be needed to obtain explosive behaviour. Of special interest are small amplitudes for which

$$c \ll \frac{b^2}{4a}. \quad (57)$$

The solutions in this case are

$$F_R \approx \begin{cases} -\frac{b}{a} \approx \frac{\Omega^2}{12|\alpha|^2}, \\ \frac{c}{b} \approx \frac{f_I^0}{\Omega}. \end{cases} \quad (58)$$

The important solution is the first one. When it is inserted in the potential $V(F_R)$ corresponding to the r.h.s. of (54),

$$V(F_R) = -c F_R - \frac{1}{2} b F_R^2 - \frac{1}{3} a F_R^3, \quad (59)$$

one obtains

$$V \approx \frac{1}{3} \frac{\Omega^6}{(24|\alpha|^2)^2}. \quad (60)$$

Since this is non-infinitesimal, there cannot be nonlinear instability with arbitrarily small initial amplitudes.

The general solution of (54) and (55) valid for all times is again given by (42)–(44), (49); and (46) also holds, but with the following new definitions of the quantities A, B, C, D :

$$\begin{aligned} A &= 8|\alpha|^2, \\ B &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ C &= -2\Omega f_I^0 + 2|\alpha|^2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1), \\ D &= (f_R^0)^2. \end{aligned} \quad (61)$$

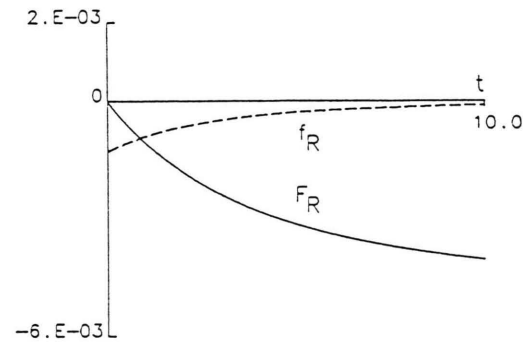
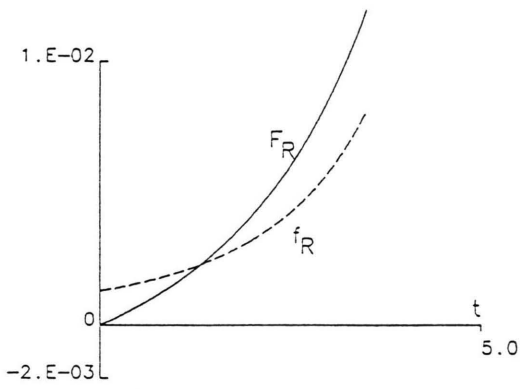
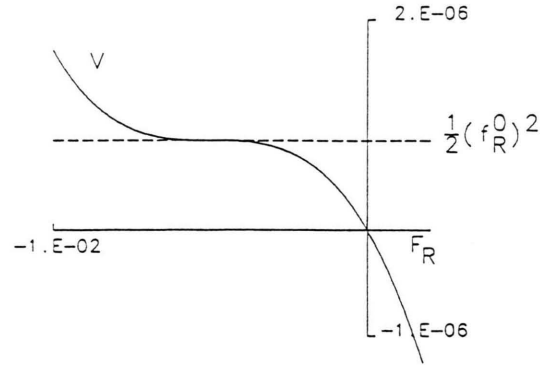
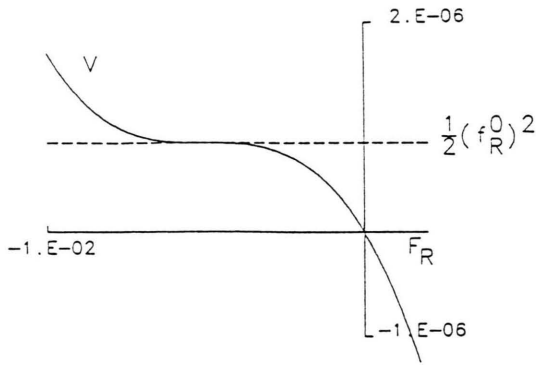
With these quantities the potential (59) can also be written as

$$V(F_R) = -\frac{1}{2}(C F_R + B F_R^2 + A F_R^3). \quad (62)$$

6. Examples

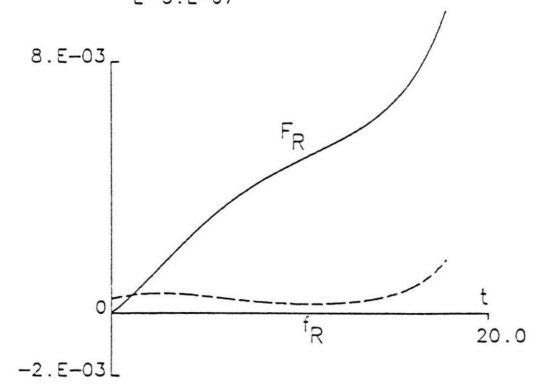
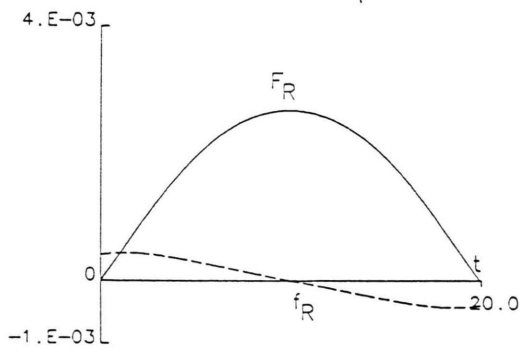
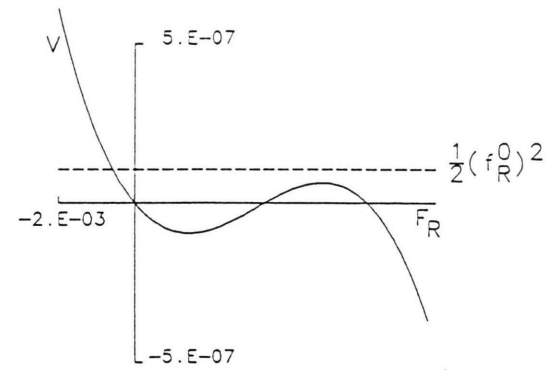
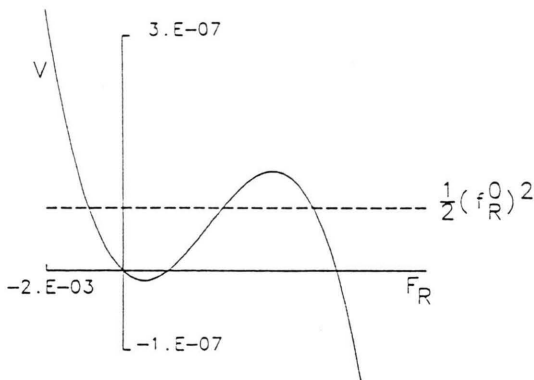
In this section some examples are given showing explosive behaviour or stability.

Insight into what is to be expected provides a more detailed discussion of the potential (62) and its negative derivative, which is the r.h.s. of (54), than the one found in the foregoing section. Figures 1a–1d may be



Figs. 2a

Figs. 2b



Figs. 2c

Figs. 2d

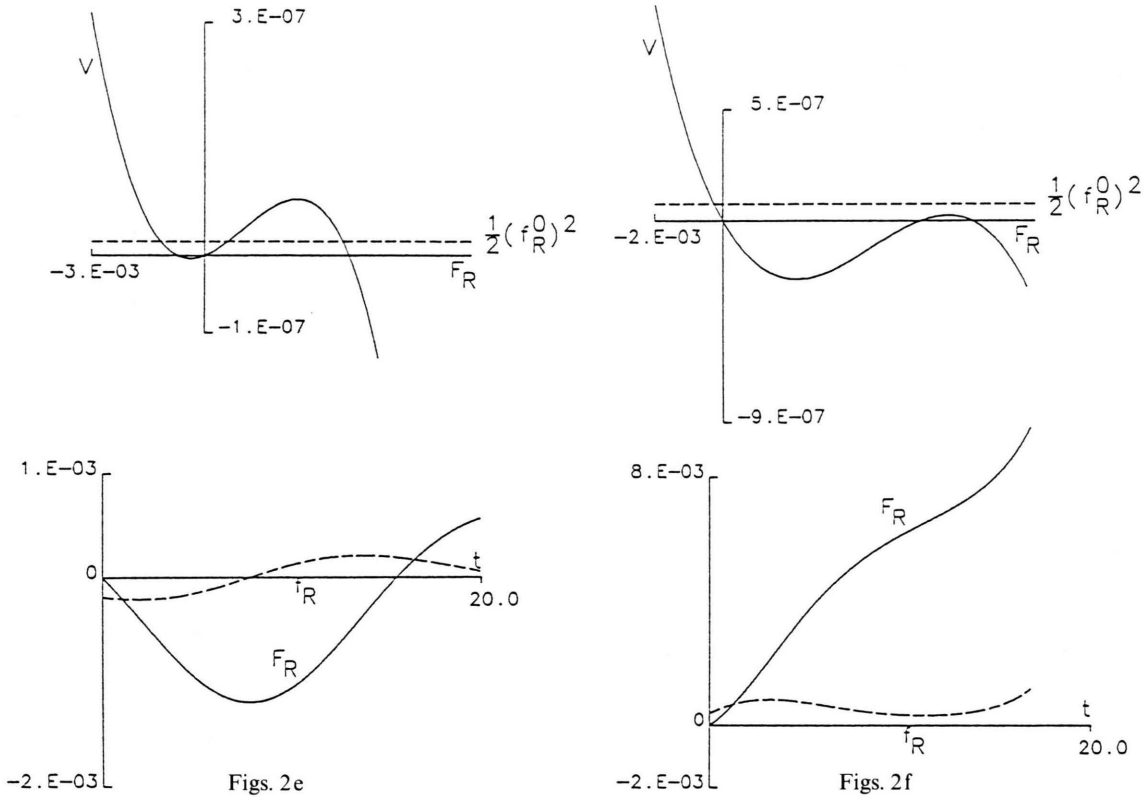


Fig. 2. Examples showing the potential $V(F_R)$ (solid line) together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ (solid line) together with $\dot{F}_R = f_R$ (dashed line).

helpful for doing this. They show typical forms of the potential $V(F_R)$.

The conditions for runaway to occur at all initial values f_R^0 are obviously that

1. equation (56) have no real zero, or
2. real zeros of (56) be at negative F_R , or
3. the maximum of $V(F_R)$ be at positive F_R , but that it be negative.

If none of these conditions is fulfilled, stable behaviour is possible if the maximum of V is at positive F_R and is positive. The final condition for runaway not to occur is then

$$\frac{1}{2}(f_R^0)^2 \leq V(F_{R_{\max}}), \quad (63)$$

where $F_{R_{\max}}$ is the position of the maximum of $V(F_R)$:

$$F_{R_{\max}} = -\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}. \quad (64)$$

Since the condition for stable behaviour being possible is a single one, it is easier to discuss this condition instead of the three conditions for runaway. For this discussion the zeros of $V(F_R)$ are of interest. They are given by

$$F_{R0} = 0, \quad F_{R\pm} = -\frac{B}{2A} \pm \sqrt{\frac{B^2}{4A^2} - \frac{C}{A}}. \quad (65)$$

One can then distinguish two cases:

1. $c < 0$,
2. $c > 0$.

For the discussion which follows, Figs. 1a to 1d may be helpful; they show typical forms of the potential $V(F_R)$.

The first case means that the derivative of $V(F_R)$ at $F_R=0$ is positive, which implies that the condition in question is fulfilled.

If one has $b > 0$ in the second case, then $V(F_R)$ is convex from above at $F_R=0$, and since the derivative

is negative, $V(F_R)$ is negative for $F_R > 0$. Hence the condition is not fulfilled.

If one has $b < 0$ in the second case, the condition is satisfied if F_{R+} is real and positive. The latter condition is automatically fulfilled with the first one. Reality requires that

$$C < \frac{B^2}{4A}. \quad (66)$$

Stable behaviour requires in addition that inequality (63) be satisfied.

When f_R^0 and f_I^0 , referring to (33), are written in the forms

$$f_I^0 = \sin\beta |\alpha| \sqrt{\lambda_1 \lambda_2 \lambda_3}, \quad f_R^0 = \cos\beta |\alpha| \sqrt{\lambda_1 \lambda_2 \lambda_3}, \quad (67)$$

$$0 \leq \beta \leq 2\pi,$$

the quantities involved in the foregoing conditions become

$$\begin{aligned} a &= 12|\alpha|^2, \\ b &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ c &= -\Omega \sin\beta |\alpha| \sqrt{\lambda_1 \lambda_2 \lambda_3} + |\alpha|^2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1), \\ A &= 8|\alpha|^2, \\ B &= -\Omega^2 + 4|\alpha|^2(\lambda_1 + \lambda_2 + \lambda_3), \\ C &= -2\Omega \mu |\alpha| \sqrt{\lambda_1 \lambda_2 \lambda_3} + 2|\alpha|^2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1), \\ D &= \cos^2\beta |\alpha|^2 \lambda_1 \lambda_2 \lambda_3. \end{aligned} \quad (68)$$

Figures 2 present typical examples; shown are the *potential* $V(F_R)$ together with $\frac{1}{2}(f_R^0)^2$ (dashed line), and $F_R(t)$ together with $\dot{F}_R = f_R$ (dashed line):

2a: $\Omega=0$; $\lambda_1=0.01$; $\lambda_2=0.01$; $\lambda_3=0.01$; $\beta=0$; *potential* like the one shown in Fig. 1 a; runaway; the solution belongs to the special class of Cherry-type solutions found for the resonant case;

2b: $\Omega=0$; $\lambda_1=0.01$; $\lambda_2=0.01$; $\lambda_3=0.01$; $\beta=\pi$; *potential* like the one shown in Fig. 1 a; $F_R(t)$ comes to a stop at the maximum of $V(F_R)$; the solution again belongs to the special class of Cherry-type solutions found for the resonant case;

2c: $\Omega=0.46$; $\lambda_1=0.01$; $\lambda_2=0.049$; $\lambda_3=0.0025$; $\beta=0.5$; *potential* like the one shown in Fig. 1 b; stable behaviour;

2d: $\Omega=0.5$; $\lambda_1=0.01$; $\lambda_2=0.0049$; $\lambda_3=0.0025$; $\beta=0$; *potential* like the one shown in Fig. 1 b; runaway;

2e: $\Omega=0.4$; $\lambda_1=0.01$; $\lambda_2=0.0049$; $\lambda_3=0.0025$; $\beta=2$; *potential* like the one shown in Fig. 1 c; stable behaviour;

2f: $\Omega=0.542$; $\lambda_1=0.01$; $\lambda_2=0.0049$; $\lambda_3=0.0025$; $\beta=-0.565$; *potential* like the one shown in Fig. 1 d; runaway.

7. Conclusions

The discussion of the complete solution of the three-oscillator case has shown that for almost all initial conditions resonance leads to an explosive behaviour. The nonlinear coupling of the three oscillators, however, allows runaway to occur in the nonresonant case as well, but the initial amplitudes must not be infinitesimally small. In a continuum theory, the three-wave coupling expression usually contains terms additional to the ones considered here. They are generally of a kind which introduces non-resonant behaviour even in the otherwise resonant case. One can speculate that their effect averages out so as to make the resonant terms dominant. This would mean that one can expect nonlinear instability rather generally, when a continuum theory allows negative energy perturbations. In [2] the same conclusion was obtained especially by referring to the original two-oscillator case of Cherry and his class of solutions. The present paper can be considered as further support for this conclusion.

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- [1] T. M. Cherry, Trans. Cambridge Philos. Soc. **23**, 199, (1925); see E. T. Whittaker, *Analytical Dynamics*, Cambridge, London 1937, Sec. 182, p. 142; – and A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University, Princeton 1947, Sect. 136, p. 101.
- [2] V. M. Dikasov, L. I. Rudakov, and D. D. Ryutov, Zh. Eksperim. i Teor. Fiz. **48**, 913 (1965). [English trans.: Soviet Phys.-JETP **21**, 608 (1966)].
- [3] R. E. Aamodt and M. L. Sloan, Phys. Rev. Lett. **19**, 1227 (1967).
- [4] B. Coppi, M. N. Rosenbluth, and R. Sudan, Ann. Phys. **55**, 207 (1968).
- [5] P. J. Morrison and M. Kotschenreuther, The Free Energy Principle, Negative Energy Modes, and Stability, DOE/ET-53088-280, IFSR # 280, Austin, Texas 78712, October 1989.
- [6] P. J. Morrison and D. Pfirsch, Phys. Fluids **B2**, 1105 (1990).